The dune size distribution and scaling relations of barchan dune fields

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Abstract Barchan dunes emerge as a collective phenomena involving the generation of thousands of them in so called barchan dune fields. By measuring the size and position of dunes in Moroccan barchan dune fields, we find that these dunes tend to distribute uniformly in space and follow an unique size distribution function. We introduce an analytical mean-field approach to show that this empirical size distribution emerges from the interplay of dune collisions and sand flux balance, the two simplest mechanisms for size selection. The analytical model also predicts a scaling relation between the fundamental macroscopic properties characterizing a dune field, namely the inter-dune spacing and the first and second moments of the dune size distribution.

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1 Introduction

A first glance over an extensive desert area shows not only that dunes are ubiquitous and present well-selected shapes, but also that they typically emerge in groups with a very well defined characteristic dune size and inter-dune spacing, forming fields of up to several thousands of dunes (see Figs. 1a-d). These observations naturally rise questions concerning the way dunes distribute throughout the deserts. What are the mechanisms of the size selection process behind such uniformity? Do the size distributions of such dune fields follow a simple unique function or do they depend on the local conditions?

Sand dunes have been intensively studied in the last years. It is now well-understood what are the fundamental laws underlying the emergence of one single barchan dune and what mechanisms maintain its shape while moving [1,2]. For instance, barchans occur in areas with unidirectional wind and low sand availability. The influence of the geographical constrains and the external physical conditions [2,3], of the dune-dune interactions [4,5,6] and even of the emergence of vegetation covers [7] in the dynamics and morphology of single dunes were quite well-established with the help of dune models [1,8,7]. There are also a few studies of entire dune fields [9,10,11,12], but a simple theoretical understanding of the size selection process within dune fields has still not been achieved.

In this work, we present a first answer to this problem. First, we show that, while a single dune is suitably characterized by its width w [2,12], an entire dune field contains dunes with different sizes following a unique distribution (Fig. 1f-g). Consequently, the corresponding average $\langle w \rangle$ and standard deviation $S = \sqrt{\langle w^2 \rangle - \langle w \rangle^2}$ are suitable properties to characterize the field. Ad-
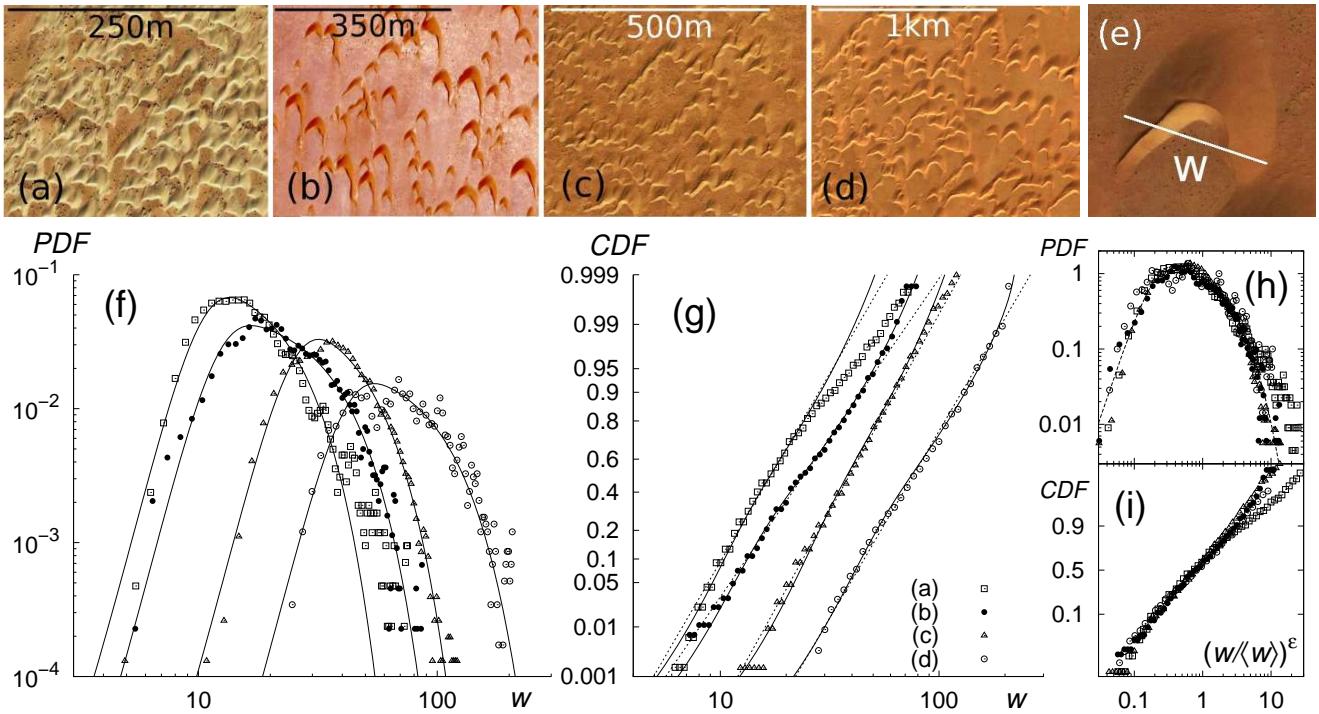
Fig. 1 (Color online) **(a-d)** Details of barchan dune fields in Morocco, Western Sahara. The number of measured dunes is 1295 (**a**), 1113 (**b**), 1947 (**c**) and 1630 (**d**), covering areas of $\sim 3, 7, 12$ and 60 km^2 and with average dune sizes of 17m, 27m, 42m and 86m respectively. The size of a barchan dune is characterized by its width w (**e**). In all pictures, the North points up. Images provided by GoogleEarth. **(f)** Probability density function (PDF) of the dune size for the measured Moroccan dune fields (symbols) and the best fit using the analytical solution (solid lines) given by Eq. (1). **(g)** Cumulative distribution function (CDF) with the analytical solution (solid lines) and the log-normal straight-line for reference (dashed lines). The relative broadness $S/\langle w \rangle$ is given by the inverse of the slope of the CDF. After rescaling the dune sizes as $(w/\langle w \rangle)^\epsilon$ with $\epsilon = 2.9 \sqrt[3]{\langle w \rangle / \langle L \rangle}$, all PDFs and CDFs in **(h)** and **(i)** respectively collapse, uncovering a scale invariance between the size distributions of different barchan dune fields. In **(h)** the analytical distribution (see text) is also shown (dashed line), as an eye-guide.

ditionally to these two quantities, we show that the inter-dune spacing L is also a property with characteristic values $\langle L \rangle$ and therefore also able to characterize the field. Second, using numerical simulations, we show that collisions between dunes play a crucial role in the selection of a characteristic dune size. Finally, from a mean-field approach that couples the effect of dune collision with that of sand flux balance, we derive the size distribution function (shown in Fig. 1f-g) and a scaling relation between the three properties of the field, $\langle L \rangle$, $\langle w \rangle$ and S.

2 Measurements of the dune size and inter-dune spacing distribution

We start by measuring the width (Fig.1e) of more than 5000 dunes composing four dune fields located in the Western Sahara (Figs. 1a-d). For all fields, the dune width distribution exhibits a unique function, apart small deviations at the extremes ($w \lesssim 10$m and $w \gtrsim$ 0.5km[12]), as shown in Fig.1f-g. This function will be derived later. The relative broadness $S/\langle w \rangle$ scales with

the relative inter-dune spacing as $\sqrt[3]{\langle L \rangle / \langle w \rangle}$, as shown in Figs. 1h-i. This scaling law relates the spatial distribution of dunes and their size distribution and will also be deduced later as a result of the size selection model we propose. Therefore, the mechanisms leading to such distributions should not depend on the absolute size and inter-dune spacing of the dunes involved. Instead, they should depend on the *relative* dune size and spacing. In other words, they should be scale invariant.

In barchan dune fields, the sand flux balance on single dunes leads to an instability in the dune size [6]. Dunes nucleate as a consequence of the sand accumulation along the field and no characteristic size emerges [6, 10]. However, the peaked size distributions in Fig. 1f are found in dune fields where sand flux balance is not the only process mediating the size of dunes. Since barchan dunes move over the field with velocities that strongly depend on the size ($v \propto 1/w$) [1], collisions are ubiquitous in such fields, turning out to be another relevant process for dune size alterations [6,12]. Therefore, the dune size distribution should be determined by the competition between the balance of sand flux on a single
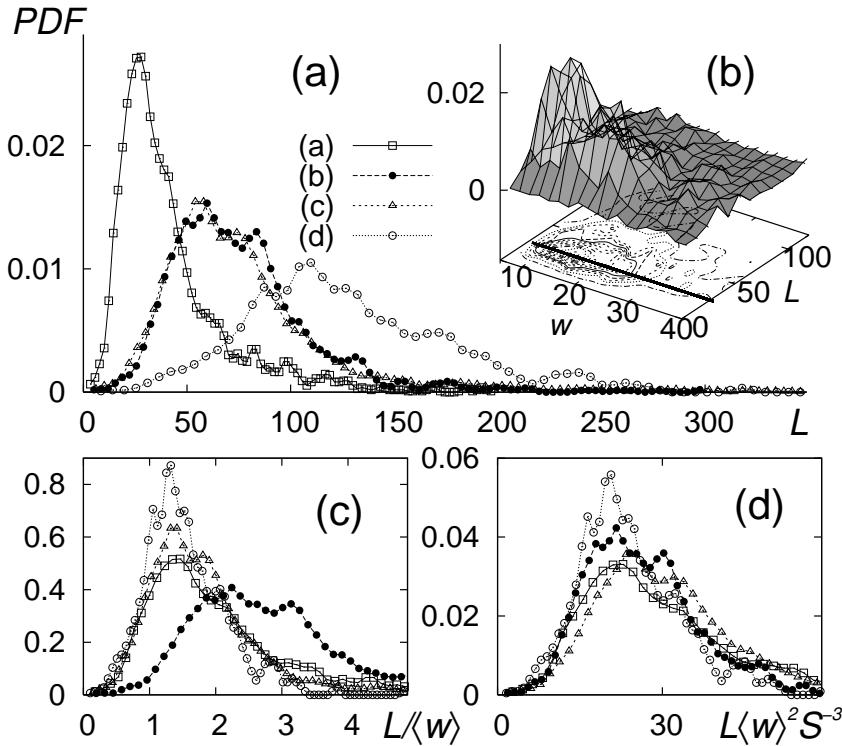
Fig. 2 **(a)** The PDF of the inter-dune spacing clearly shows a characteristic value $\langle L \rangle$ for each dune field depicted in Fig.1(a-d). The inter-dune spacing L around a given dune is defined as the square root of the empty area of a polygon formed by the centers of the nearest dunes, one in each quadrant of the Cartesian frame reference centered at the dune. **(b)** PDF of L as a function of the dune width w for the first dune field **a**. From the contour plot, one defines the characteristic inter-dune spacing $\langle L \rangle$ (solid line), taken as the average over the highest frequency region, which is independent of w (see text). **(c)** After rescaling L by the mean dune size $\langle w \rangle$, not all PDFs peak in the same relative inter-dune spacing $\langle L \rangle / \langle w \rangle$. **(d)** However, the curves collapse after rescaling L by the expression $S^3/\langle w \rangle^2$, where S represents the standard deviation of the size distribution (see text).

dune and the collisions between neighboring dunes [5, 6,10]. Recently a third size selection mechanism was discovered, that involves the calving of large dunes due to wind fluctuations [12,13]. This is a complex scale dependent process, relevant for fields with large dunes and should not be responsible for the distributions here addressed. Therefore, we will not consider it.

Two important aspects must now be addressed to proceed further. First, we notice that in the absence of motion, i.e. in static fields where no collisions can occur, one dune grows only if its neighboring dunes shrink [14], due to sand flux balance and mass conservation. Consequently, if collisions do not occur the inter-dune spacing L between neighboring dunes would scale with the dune size. Our empirical data however, shows a rather different behavior. For each dune field in Figs. 1a-d, the inter-dune spacing L between each dune and its neighbors as a function of the width w, distributes parallel to the w-axis (Fig. 2b). Thus, contrary to the situation without collisions, here the inter-dune spacing can be taken as its characteristic value, say $\langle L \rangle$, as shown in Fig. 2a and 2b. This empirical result is a clear sign of

a richer internal dynamics in dune fields where collisions play an important role. Indeed, due to collisions, small dunes are continuously emerging from larger ones [12,13], destroying any simple correlation between dune size and inter-dune spacing, and therefore leading to the observed spatial uniformity.

The second aspect is that $\langle L \rangle$ does not follow a simple scaling with the average dune size $\langle w \rangle$ of the field (Fig. 2c) but a more complex one (as $S^3/\langle w \rangle^2$, Fig. 2d) which also involves the standard deviation S of the size distribution. As will be shown later, this scaling naturally emerges from the coupling of binary dune collisions and sand flux balance in the size selection process.

3 Binary collision dynamics

With these two empirical findings we proceed by studying the effect of collisions alone in the size distribution of a dune field. Using an established dune model [1,8,5], we simulate ideal binary collisions, under open boundary conditions and constant wind (Fig.3a), extracting a simple collision rule, i.e. a phenomenological function

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that relates the relative dune size r_f after the collision between two dunes, and the corresponding initial relative size r_i and offset θ_i [15]. In accordance to recent underwater experiments [6] and our simulations, for most initial conditions the number of dunes is conserved and the total sand volume change is negligible. Furthermore, as seen in Fig. 3b, in most of the cases the collision increases the relative dune size ($r_f > r_i$), redistributing sand from large to small dunes, in sharp contrast to the flux balance, which accumulates sand on large dunes [15].
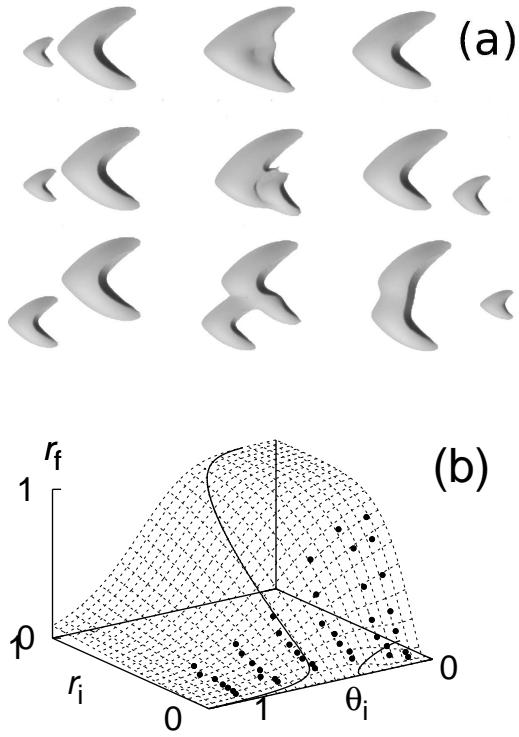
Fig. 3 (a) Different outcomes of simulated binary collisions, from coalescence (when both dunes merge) on top, to a situation where the volume of the smaller dune increases (decreases) after the collision (middle and bottom sections, respectively). **(b)** Collision rule for binary collisions that conserve the number of dunes. Dots represent numerical simulations and dashed lines the corresponding surface fit $r_f(r_i, \theta_i)$. The curve $r_f(r_i, \theta_i) = r_i$ (solid line) separates two regimes: one with $r_f > r_i$, where collisions redistribute sand and another with $r_f < r_i$ due to accumulation of sand.

With this collision rule we study the influence of collisions on the dune size distribution neglecting sand exchange through sand flux between them. A large sampling statistics of about 10 000 dunes is considered, where we assume that each pair of dunes collides with the same fixed probability following the collision rule

introduced above (Fig. 3b). This assumption does not consider the spatial distribution of dunes and thus we neglect the spatial correlations between dune sizes and positions. As a result of binary collisions, arbitrary initial size distributions converge to a stationary Gaussian distribution $P_{col}(w)$, illustrated in Fig. 4a. We find that the relative standard deviation $\sigma_{col}/\langle w \rangle_{col}$ of such distribution, where $\langle w \rangle_{col}$ is the mean size, is constant for different initial conditions (Fig. 4b) and its value is fixed by the parameters of the dune model, which were chosen to reproduce the morphology of Moroccan dunes [5,16], which in turn determine the dune morphology and thus the phenomenological collision rule [15]. The constant value of the relative standard deviation is consequence of the scale invariance of our collision dynamics that depends on the relative dune size instead of the absolute size. On the contrary, with changing wind conditions, the dune scale may influence the collision outcome and also several dunes may emerge from a single collision [13].

Until now, two preliminary conclusions can be stated. First, the measured barchan dunes are approximately uniformly placed over the deserts, with characteristic values of L, and the different dune fields follow a common size distribution with a simple scaling with the mean dune size. Second, neither sand flux alone does contribute to the size selection [6,10] nor collisions alone can be responsible for the skewed measured non-Gaussian distributions (Fig.1). What results from the interplay between both mechanisms will be now derived from a mean-field approach, leading to a master equation for the size distribution $P(w,t)$ and to a simple scaling between all the three properties, $\langle w \rangle$, S and $\langle L \rangle$.

4 Master equation

When only ideal collisions occur, Fig. 4c shows that both, the mean $\langle w \rangle$ and the standard deviation σ of the size distribution exponentially relax toward their respective equilibrium values, $\langle w \rangle_{col}$ and σ_{col}. By using the definition $\langle w \rangle(t) \equiv \int dw \; w \; P(w,t)$ and the exponential relaxation $d\langle w \rangle/dt = (\langle w \rangle_{col} - \langle w \rangle)/t_c$, the size distribution $P(w,t)$ obeys in first approximation the dynamical equation $dP/dt = (P_{col} - P)/t_c$, where t_c is the characteristic relaxation time (in units of number of collisions per dune) of $P(w)$ towards the equilibrium Gaussian distribution P_{col} when only collisions are considered.

When both collisions and sand flux balance are considered, the total temporal derivative of $P(w,t)$ has now two separated terms, the partial temporal derivative $\partial P/\partial t$ and the term arising from the volume change rate $\dot{V} = \dot{w}dV/dw$, due to flux balance: $\dot{V}\partial P/\partial V =$
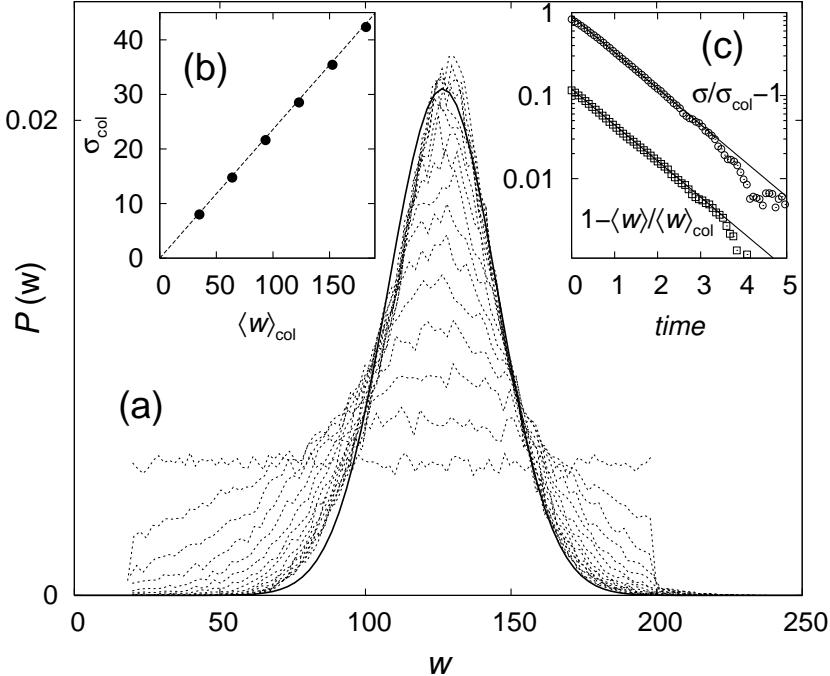
Fig. 4 **(a)** Snap-shots of the evolution due to binary collisions of a uniform dune size distribution (dashed-lines) towards a Gaussian (solid line). **(b)** Linear relation between the two first moments of the observed Gaussian, for different initial volumes of sand available, yielding a constant relative standard deviation $\sigma_{col}/\langle w\rangle_{col}$. **(c)** The first two moments $\langle w\rangle$ (squares) and σ (circles), exponentially relax toward their equilibrium values $\langle w\rangle_{col}$ and σ_{col} (see text). Time units are number of collisions per dune.

$\dot{w}\partial P/\partial w$. Assuming the existence of a steady state and using the empirical fact that $V \propto w^3$ and $\dot{V} \propto Qw$, with Q denoting the saturated sand flux over a flat bed [12], the master-equation yields

$$\frac{t_c}{t_s}\frac{\partial P(w)}{\partial w} = \frac{w}{\langle w\rangle_{col}^2}\left(P_{col}(w) - P(w)\right), \qquad (1)$$

where there are three parameters, namely the relative deviation $\sigma_{col}/\langle w\rangle_{col}$ of P_{col}, determined by the collision model, the characteristic size $\langle w\rangle_{col}$ and the ratio t_c/t_s. Time t_s is the characteristic time associated to the change rate of the dune size due to the sand flux balance, defined as $t_s = \alpha\langle w\rangle_{col}^2/Q$ with α as a constant. From Eq. (1) one concludes that when collisions dominate in the selection of dune sizes, $t_c \ll t_s$ and consequently the distribution converges to the Gaussian P_{col}. When the opposite occurs, with the sand flux balance being the relevant process, $P(w)$ deviates from $P_{col}(w)$ the more the larger t_c/t_s is.

The solution of Eq. (1) is plotted in Fig. 1f and 1g (solid lines) for each dune field, with $\langle w\rangle_{col}$ and t_c/t_s as fit parameters. The value of σ_{col} is taken from Fig. 4b. Apart extreme points, the solution fits well the empirical distributions, with first and second moments rea-

sonable approximated as

$$\langle w\rangle \simeq 0.8\langle w\rangle_{col}(\sqrt{t_c/t_s} + 1) \qquad (2a)$$
$$S/\langle w\rangle \simeq 0.62\sqrt{t_c/t_s}/(\sqrt{t_c/t_s} + 1) \qquad (2b)$$

in the range $t_s < 5t_c$. From Eqs. (2) one sees that the characteristic size $\langle w\rangle_{col}$ determined by the collisions dynamics is in fact the only characteristic size in the system. Moreover, Eq. (2b) shows that the relative standard deviation $S/\langle w\rangle$ is given by the ratio t_c/t_s and thus describes a measure for the competition between sand flux balance (t_s) and collision (t_c) processes for the dune size selection. For instance, dune field in Fig. 1b has a large value $t_c/t_s = 4.3$, indicating that the sand flux balance is the most important size selection process, while the dune field in Fig. 1c has $t_c/t_s = 1.7$ indicating more relevance from collision processes.

Furthermore, the ratio t_c/t_s is not an independent parameter since t_c must be proportional to the collision time t_{col}, defined as the average time for two dunes to collide. This collision time is determined as the quotient between the inter-dune spacing $\langle L\rangle$ and the average relative velocity between two dunes, $\langle v\rangle_r \propto \int_0^\infty dw_1 \int_{w_1}^\infty dw_2 P(w_1)P(w_2)(v(w_1) - v(w_2))$. Since the dune velocity follows $v \propto Q/w$ and, within some wide range of sizes, $P(w)$ can be well approximated by a log-normal distribution (see Figs. 1f and 1g), one obtains